MATH2048 Honours Linear Algebra II

Midterm Examination 1

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let $V = M_{2\times 2}(\mathbb{R})$. Define $T : V \to \mathbb{R}$ by $T(A) = A_{11} - A_{22}$ and $U : P_1(\mathbb{R}) \to V$ by $U(p) = \begin{pmatrix} 0 & p(0) \\ -p(0) & p(1) \end{pmatrix}$.

   (a) Find a basis β for $N(T)$ and a basis γ for $R(U)$.

   (b) Find the dimensions of $N(T)$, $R(U)$, $N(T) \cap R(U)$ and $N(T) + R(U)$.

   Proof.

   (a) The nullspace of T, denoted $N(T)$, consists of all matrices $A \in M_{2\times 2}(\mathbb{R})$ such that $T(A) = 0$, i.e., $A_{11} = A_{22}$. A basis β for this space can be given by

   $$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

   The range of U, denoted $R(U)$, consists of all matrices in $M_{2\times 2}(\mathbb{R})$ of the form $U(p)$ for some $p \in P_1(\mathbb{R})$. It can be seen that it only depends on the values of $p(0)$ and $p(1)$, and that these can be any real numbers. Therefore, a basis γ for this space can be given by

   $$\gamma = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

   (b) i. The dimension of $N(T)$ is the number of vectors in a basis for $N(T)$, which is 3.

   ii. Similar, the dimension of $R(U)$ is 2.

   iii. Note that the sum $N(T) + R(U)$ includes $e_{11} + e_{22}$, e_{12}, e_{21}, and e_{22}. This spans the entire space $M_{2\times 2}(\mathbb{R})$, and hence the dimension of the sum $\dim(N(T) + R(U)) = 4$.

   iv. Now, using the formula $\dim(N(T) \cap R(U)) + \dim(N(T) + R(U)) = \dim(N(T)) + \dim(R(U))$, we can find the dimension of the intersection $N(T) \cap R(U)$. This gives us $\dim(N(T) \cap R(U)) = \dim(N(T)) + \dim(R(U)) - \dim(N(T) + R(U)) = 3 + 2 - 4 = 1$.

   $\square$

2. Consider the linear operator T defined by

$$\begin{aligned} T : M_{2\times 2}(\mathbb{R}) &\to M_{2\times 2}(\mathbb{R}) \\ A &\mapsto A - A^T \end{aligned}$$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the standard ordered basis for $M_{2\times 2}(\mathbb{R})$.

(a) Find an ordered basis γ_1 for $N(T)$ and an ordered basis γ_2 for $R(T)$. Show that $\gamma = \gamma_1 \cup \gamma_2$ is an ordered basis for $M_{2\times 2}(\mathbb{R})$.

(b) With the γ that you found in (a), find the matrices $A = [T]_\gamma$ and $B = [I_{M_{2\times 2}(\mathbb{R})}]_\gamma^\beta$. Represent $[T]_\beta$ by A and B using change of coordinates (No need to do the computation).

Proof.

(a) i. The null space of T, $N(T)$, consists of all matrices $A \in M_{2\times 2}(\mathbb{R})$ such that $T(A) = A - A^T = 0$, i.e., $A = A^T$. These are the symmetric matrices. An ordered basis γ_1 for this space can be given by

$$\gamma_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

ii. The range of T, $R(T)$, consists of all matrices of the form $T(A) = A - A^T$ for some $A \in M_{2\times 2}(\mathbb{R})$. These are the antisymmetric matrices. An ordered basis γ_2 for this space can be given by

$$\gamma_2 = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

iii. The union of γ_1 and γ_2 is

$$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

This set of matrices spans $M_{2\times 2}(\mathbb{R})$ and is linearly independent, and so is an ordered basis for $M_{2\times 2}(\mathbb{R})$.

(b) i. Since T sends symmetric matrices to the zero matrix and antisymmetric matrices to twice themselves, we have

$$A = [T]_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

ii. To find the matrix $B = [I_{M_{2\times 2}(\mathbb{R})}]_\gamma^\beta$, we note that the identity map sends each basis vector to itself.

$$B = [I_{M_{2\times 2}(\mathbb{R})}]_\gamma^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

iii. Given the matrices A and B, we can represent the matrix of the transformation T in the β basis, $[T]_\beta$, as

$$[T]_\beta = BAB^{-1}.$$

$\square$

3. Consider the linear transformation

$$\begin{aligned} T : P(\mathbb{R}) &\to P(\mathbb{R}) \\ p(x) &\mapsto p(x+1) \end{aligned}$$

Note that $P(\mathbb{R}) = \bigcup_{n=0}^{\infty} P_n(\mathbb{R})$. Let $T|_{P_n(\mathbb{R})} : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the restriction of T on $P_n(\mathbb{R})$. Let β_n be the standard ordered basis for $P_n(\mathbb{R})$ for non-negative integer n.

(a) Prove that $T(\beta_n)$ is a basis for $P_n(\mathbb{R})$, that is $T|_{P_n(\mathbb{R})}$ is onto. Deduce that T is onto. (Hint: The Binomial Theorem $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$.)

(b) Use (a) to show that $T|_{P_n(\mathbb{R})}$ is one-to-one. Deduce that T is one-to-one.

Proof.

(a) To prove that $T(\beta_n)$ is a basis for $P_n(\mathbb{R})$, we first note that the standard ordered basis for $P_n(\mathbb{R})$ is $\beta_n = \{1, x, x^2, ..., x^n\}$.
Under the linear transformation T, each basis vector x^i is mapped to $(x+1)^i$. Therefore, $T(\beta_n)$ is given by $\{1, (x+1), (x+1)^2, ..., (x+1)^n\}$.
Next, we consider the matrix representation of T with respect to the basis β_n, denoted as $[T]_{\beta_n}$. The i-th column of $[T]_{\beta_n}$ is the coordinates of $T(x^i)$ with respect to β_n.
Using the Binomial theorem, the i-th column of $[T]_{\beta_n}$ can be written as a vector whose j-th entry is $\binom{i}{j}$, for $j = 0, ..., i$, and is 0 for $j = i+1, ..., n$. Therefore, $[T]_{\beta_n}$ is an upper triangular matrix with 1s on the diagonal, which means it is invertible.
Since the transformation matrix is invertible, it means that the columns of the matrix, which correspond to the images of the basis vectors under T, are linearly independent. Therefore, $T(\beta_n)$ forms a basis for $P_n(\mathbb{R})$.
Finally, since $P(\mathbb{R}) = \bigcup_{n=0}^{\infty} P_n(\mathbb{R})$, and $T|_{P_n(\mathbb{R})}$ is onto for each n, T is onto on $P(\mathbb{R})$.

(b) Note that $T|_{P_n(\mathbb{R})}$ is a surjective linear map from a finite dimensional space to itself, therefore $T|_{P_n(\mathbb{R})}$ is one-to-one.
[Another approach: To show that $T|_{P_n(\mathbb{R})}$ is one-to-one, we need to show that if $T(p(x)) = T(q(x))$, then $p(x) = q(x)$. Suppose $T(p(x)) = T(q(x))$, then $p(x+1) = q(x+1)$. This means that the polynomials $p(x)$ and $q(x)$ are equal for all $x \in \mathbb{R}$, and thus $p(x) = q(x)$.]
Then, $T|_{P_n(\mathbb{R})}$ is one-to-one. For $p, q \in P(\mathbb{R})$. Suppose $T(p) = T(q)$. Note that there is some n such that $p, q \in P_n(\mathbb{R})$, and $T|_{P_n(\mathbb{R})}$ is one-to-one. Then $p = q$. Therefore, T is one-to-one on $P(\mathbb{R})$.

$\square$

4. Let $C^\infty(\mathbb{R})$ be the vector space of all smooth real functions (infinitely differentiable) over $\mathbb{R}$. Let V be the subspace of $C^\infty(\mathbb{R})$ defined by:

$$V = \{f \in C^\infty(\mathbb{R}) : f(0) = f'(0) = ... = f^{(n)}(0) = 0\},$$

where $f^{(j)}$ denotes the j-th derivative of f.

(a) Define $\Psi : C^\infty(\mathbb{R}) \to \mathbb{R}^{n+1}$ by $\Psi(f) = (f(0), f'(0), ..., f^{(n)}(0))$. Show that Ψ is onto.

(b) Define $\tilde{\Psi} : C^\infty(\mathbb{R})/V \to \mathbb{R}^{n+1}$ by $\tilde{\Psi}(f+V) = \Psi(f)$. Use (a) to show that $\tilde{\Psi}$ is an isomorphism, i.e. well-defined, linear and bijective.

Proof.

(a) We need to show that Ψ is onto, which means that for every element $(r_0, r_1, ..., r_n) \in \mathbb{R}^{n+1}$, there is some $f \in C^\infty(\mathbb{R})$ such that $\Psi(f) = (r_0, r_1, ..., r_n)$. Consider the function $f(x) = r_0 + r_1 x + r_2 \frac{x^2}{2} + \ldots + r_n \frac{x^n}{n!}$. Then, for each $j \in \{0, 1, ..., n\}$, we have $f^{(j)}(0) = r_j$, and so $\Psi(f) = (r_0, r_1, ..., r_n)$. Thus, Ψ is onto.

(b) To show that $\tilde{\Psi}$ is an isomorphism, we need to prove it is well-defined, linear, and bijective:

i. *Well-defined:* If $f + V = g + V$ for some $f, g \in C^\infty(\mathbb{R})$, then $f - g \in V$, so $f(0) = g(0)$, $f'(0) = g'(0)$, ..., $f^{(n)}(0) = g^{(n)}(0)$. Hence, $\Psi(f) = \Psi(g)$, so $\tilde{\Psi}(f+V) = \tilde{\Psi}(g+V)$, and $\tilde{\Psi}$ is well-defined.

ii. *Linearity:* This follows immediately from the linearity of Ψ and the definition of vector addition and scalar multiplication in the quotient space $C^\infty(\mathbb{R})/V$.

iii. *Bijectivity:* By definition, the kernel of $\tilde{\Psi}$ is V, so $\tilde{\Psi}$ is injective. And since we've shown in part (a) that Ψ is onto, so too is $\tilde{\Psi}$. Hence, $\tilde{\Psi}$ is bijective.

□

5. Let U be a non-zero subspace of an infinite dimensional vector space V over F. Let $L \subset U$ be a basis of U. Using Zorn's lemma, show that L can be extended to a basis of V. Deduce that there exists a subspace W of V such that $V = U \oplus W$. Please explain your answer with all the details.

Proof. Let U be a non-zero subspace of an infinite dimensional vector space V over a field F. Let $L \subset U$ be a basis for U.

We aim to show that L can be extended to a basis of V using Zorn's lemma. To do so, consider the set S of all linearly independent subsets of V that contain L. This set S is non-empty since it contains L, and it is partially ordered by inclusion.

Given a chain C in S (a totally ordered subset of S), we can show that it has an upper bound in S. Take $L' = \bigcup C$. As each set in C is linearly independent and each set in the chain contains L, L' is linearly independent and contains L, so $L' \in S$. Therefore, L' is an upper bound for the chain C in S.

By Zorn's lemma, S has a maximal element, say B. If B is not a basis for V, then it must not span V. There exists a vector $v \in V$ not in the span of B. Add v to B, to get a larger linearly independent set contradicting the maximality of B. Hence, B is a basis for V.

Finally, let $W = B \setminus L$. Then W is a subspace of V that is disjoint from U (since L is a basis for U), and $V = U \oplus W$. This is because any vector $v \in V$ can be written uniquely as $v = u + w$ for some $u \in U$ and $w \in W$. Here, the uniqueness follows from the fact that $B = L \cup W$ is a basis for V. □